

Lie point symmetries of a general class of PDEs: The heat equation

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Abstract

We give two theorems which show that the Lie point and the Noether symmetries of a second-order ordinary differential equation of the form $\frac{D}{Ds} \left(\frac{Dx^i(s)}{Ds} \right) = F(x^i(s), \dot{x}^i(s))$ are subalgebras of the special projective and the homothetic algebra of the space respectively. We examine the possible extension of this result to partial differential equations (PDE) of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$ where $u(x^i)$ and u_{ij} stands for the second partial derivative. We find that if the coefficients A^{ij} are independent of $u(x^i)$ then the Lie point symmetries of the PDE form a subgroup of the conformal symmetries of the metric defined by the coefficients A^{ij} . We specialize the study to linear forms of $F(x^i, u, u_i)$ and write the Lie symmetry conditions for this case. We apply this result to two cases. The wave equation in an inhomogeneous medium for which we derive the Lie symmetry vectors and check our results with those in the literature. Subsequently we consider the heat equation with a flux in an n -dimensional Riemannian space and show that the Lie symmetry algebra is a subalgebra of the homothetic algebra of the space. We discuss this result in the case of de Sitter space time and in flat space.

Keywords: Lie point symmetries, Homothetic motions, Partial Differential Equations, Heat Equation.

PACS - numbers: 2.40.Hw, 4.20.-q, 4.20.Jb, 04.20.Me, 03.20.+i, 02.40.Ky

1 Introduction

A Lie point symmetry of an ordinary differential equation (ODE) is a point transformation in the space of variables which preserves the set of solutions of the ODE [1, 2, 3]. If we look at these solutions as curves in the space of variables, then we may equivalently consider a Lie point symmetry as a point transformation which preserves the set of solution curves. Applying this observation to the geodesic curves in a Riemannian (affine) space, we infer that the Lie point symmetries of the geodesic equations in any Riemannian (affine) space are the automorphisms which preserve the set of these curves. However, it is known by Differential Geometry that the point transformations of a Riemannian (affine) space which preserve the set of geodesics are the projective transformations. Therefore it is reasonable to expect a correspondence between the Lie symmetries of the geodesic equations and the projective algebra of the space.

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The equation of geodesics in an arbitrary coordinate frame is a second-order ODE of the form

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k + F(x^i, \dot{x}^j) = 0 \quad (1)$$

where $F(x^i, \dot{x}^j)$ is an arbitrary function of its arguments and the functions Γ_{jk}^i are the connection coefficients of the space.¹ Equivalently equation (1) is the equation of motion of a dynamical system moving in a Riemannian (affine) space under the action of a velocity dependent force. According to the above argument we expect that the Lie symmetries of the ODE (1) for a given function $F(x^i, \dot{x}^j)$ are a subalgebra of the projective algebra of the space. This subalgebra is selected by means of certain constraint conditions which will involve geometric quantities of the space and the function $F(x^i, \dot{x}^j)$. This approach is not new and similar considerations can be found in [4, 5, 6, 7, 8, 9].

The determination of the Lie point symmetries of a given system of ODEs consists of two steps (a) the determination of the conditions which the components of the Lie symmetry vectors must satisfy and (b) the solution of the system of these conditions. Step (a) is formal and it is outlined in e.g. [1, 2, 3]. The second step is the key one and, for example, in higher dimensions where one has a large number of simultaneous equations the solution can be quite involved. However, if we express the system of Lie symmetry conditions of (1) in a Riemannian space in terms of collineation (i.e. symmetry) conditions of the metric, then the determination of Lie symmetries is transferred to the geometric problem of determining the generators of the projective group of the metric. In this field there is a vast amount of work that is already done to be used. Indeed the projective symmetries are already known in many cases or they can be determined by existing general theorems of Differential Geometry. For example the projective algebra and all its subalgebras are known for the spaces of constant curvature [10] and in particular for the flat spaces. This implies, for example, that the Lie symmetries of *all* Newtonian dynamical systems are "known" and the same applies to dynamical systems in Special Relativity!

In this work we state a theorem which establishes the exact relation between the projective algebra of the space and the Lie symmetry algebra of (1), assuming that the function F depends only on the coordinates, i.e. $F(x^i)$.

What has been said for the Lie point symmetries of (1) applies also to Noether symmetries (provided (1) follows from a Lagrangian). The Noether symmetries are Lie point symmetries which satisfy the constraint

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt}. \quad (2)$$

Noether symmetries form a closed subalgebra of the Lie symmetries algebra. In accordance to the above, this implies that the Noether symmetries will be related with a subalgebra of the projection algebra of the space where 'motion' occurs. As it will be shown, this subalgebra is contained in the homothetic algebra of the space.

As it is well known, each Noether point symmetry is associated conserved current (i.e. first integral), hence the above imply that the (standard) conserved quantities of a dynamical system depend on the space it moves and the type of force $F(x^i, \dot{x}^j)$ which modulates the motion. In particular, in 'free fall', i.e. in the case $F(x^i, \dot{x}^j) = 0$ the *geometry* of the space is the sole factor which determines the (standard) first integrals of motion. This conclusion is by no means trivial and shows the deep relation between Geometry and Physics!

¹This point of view can be generalized to the general second order ODE provided the functions Γ_{jk}^i can be identified with the connection coefficients of a metric.

A natural question which arises is the following:

To what extent this correspondence of Lie/Noether symmetries of second order ODES of the form (1), with the collineations of the space, is extendable to partial differential equations of second-order of a similar form?

Obviously, a global answer to this question is not possible. However, it can be shown that for many interesting PDEs the Lie symmetries are indeed obtained from the collineations of the metric. Pioneering work in this direction is the work of Ibragimov [11]. Recently, Bozhkov et al. [12] studied the Lie and the Noether symmetries of the Poisson equation and shown that the Lie symmetries of the Poisson PDE are generated from the conformal algebra of the metric.

In the present work we show that for a general class of PDEs of second-order, there is a close relation between the Lie symmetries and the conformal algebra of the space. Subsequently we apply these results to a number of interesting PDEs and regain existing results in a unified manner. As a new application, we determine the Lie symmetries of the heat equation with a flux in a n -dimensional Riemannian space.

The structure of the paper is as follows. In Section 2 we review briefly the collineations in a Riemannian space. In Section 3 we consider the equation of geodesics in an affine space and determine the Lie symmetry conditions in covariant form. We find that the major symmetry condition relates the Lie symmetries with the special projective algebra of the space. A similar result has been obtained previously in [4] using the bundle formulation of second order ODEs.

In Section 4 we solve, in a concise manner, the symmetry conditions and state Theorem 1 which gives the Lie symmetry vectors in terms of the collineations of the metric and Theorem 2 which gives the Noether point symmetries in terms of the homothetic algebra of the metric.

In Section 5 we consider the PDEs of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$ and derive the Lie symmetry conditions. We show that for these PDEs the $\xi^i(x^j)$ provided $A^{ij} \neq 0$ and if $A^{ij}_{,u} = 0$ then $\xi^i(x^j)\partial_i$ is a Conformal Killing vector of the metric A^{ij} .

In Section 6 we consider a linear form of $F(x^i, u, u_i)$ and determine the Lie symmetry conditions in geometric form. In section 7 we apply the results of section 6 to the wave equation in a two dimensional inhomogeneous medium and to the heat equation with flux in a n -dimensional Riemannian space. In the latter case, it is shown that the Lie symmetry vectors are obtained from the homothetic algebra of the A^{ij} metric. A similar result has been found for the Poisson equation [12].

Finally in Section 8 we comment on the results and point out possible directions for future research.

2 Collineations of Riemannian spaces

A collineation in a Riemannian space is a vector field \mathbf{X} which satisfies an equation of the form

$$\mathcal{L}_X \mathbf{A} = \mathbf{B} \quad (3)$$

where \mathcal{L}_X denotes Lie derivative, \mathbf{A} is a geometric object (not necessarily a tensor) defined in terms of the metric and its derivatives (e.g. connection, Ricci tensor, curvature tensor etc.) and \mathbf{B} is an arbitrary tensor with the same tensor indices as \mathbf{A} . The collineations in a Riemannian space have been classified by Katzin et al. [13]. In the following we use only certain collineations.

A conformal Killing vector (CKV) is defined by the relation

$$\mathcal{L}_X g_{ij} = 2\psi(x^k) g_{ij}. \quad (4)$$

If $\psi = 0$, \mathbf{X} is called a Killing vector (KV). If ψ is a nonvanishing constant then \mathbf{X} is a homothetic vector (HV) and if $\psi_{;ij} = 0$, \mathbf{X} is a special conformal Killing vector (SCKV). A CKV is called proper if it is not a KV, a HV or a SCKV.

A Projective collineation (PC) is defined by the equation

$$\mathcal{L}_X \Gamma_{jk}^i = 2\phi_{(j} \delta_{k)}^i. \quad (5)$$

If $\phi = 0$, the PC is called an affine collineation (AC) and if $\phi_{;ij} = 0$, a special projective collineation (SPC). A proper PC is a PC which is not an AC, HV or KV or SPC. The PCs form a Lie algebra whose ACs, HV and KVs form subalgebras. It has been shown that if a metric admits a SCKV, then also admits a SPC, a gradient HV and a gradient KV [14].

In the following we shall need the symmetry algebra of spaces of constant curvature. In [10] it has been shown that the PCs of a space of constant nonvanishing curvature consist of proper PCs and KVs only and if the space is flat then the algebra of the PCs consists of KVs/HV/ACs and SPCs.

In particular, for the Euclidian space E^n the projection algebra consists of the vectors of in Table 1.

Table 1: Collineations of Euclidean space E^n , $I, J = 1, 2, \dots, n$.

Collineation	Gradient	Nongradient
Killing vectors (KV)	$\mathbf{S}_I = \delta_I^i \partial_i$	$\mathbf{X}_{IJ} = \delta_{[I}^j \delta_{J]}^i x_j \partial_i$
Homothetic vector (HV)	$\mathbf{H} = x^i \partial_i$	
Affine Collineation (AC)	$\mathbf{A}_{II} = x_I \delta_I^i \partial_i$	$\mathbf{A}_{IJ} = x_J \delta_I^i \partial_i$
Special Projective collineation (SPC)		$\mathbf{P}_I = S_I \mathbf{H}$.

3 The Lie point symmetry conditions in an affine space

We consider the system of ODEs:

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k + \sum_{m=0}^n P_{j_1 \dots j_m}^i \dot{x}^{j_1} \dots \dot{x}^{j_m} = 0 \quad (6)$$

where Γ_{jk}^i are the connection coefficients of the space and $P_{j_1 \dots j_m}^i(t, x^i)$ are smooth polynomials completely symmetric in the lower indices and derive the Lie point symmetry conditions in geometric form using the standard approach. Equation (6) is quite general and covers most of the standard cases, autonomous and non autonomous, and in particular equation (1). Furthermore because the Γ_{jk}^i 's are not assumed to be symmetric, the results are valid in a space with torsion. Obviously they hold in a Riemannian space provided that the connection coefficients are given in terms of the Christoffel symbols.

The detailed calculation has been given in [9] and shall not be repeated here. In the following we summarize these results.

The terms $\dot{x}^{j_1} \dots \dot{x}^{j_m}$ for $m \leq 4$ give the equations:

$$L_\eta P^i + 2\xi_{,t} P^i + \xi P^i_{,t} + \eta^i_{,tt} + \eta^j_{,t} P^i_{,j} = 0 \quad (7)$$

$$L_\eta P^i_j + \xi_{,t} P^i_j + \xi P^i_{j,t} + (\xi_{,k} \delta^i_j + 2\xi_{,j} \delta^i_k) P^k + 2\eta^i_{,t|j} - \xi_{,tt} \delta^i_k + 2\eta^k_{,t} P^i_{,jk} = 0 \quad (8)$$

$$L_\eta P^i_{jk} + L_\eta \Gamma^i_{jk} + (\xi_{,d} \delta^i_{(k} + \xi_{,(k} \delta^i_{|d|})} P^d_{,j}) + \xi P^i_{,kj,t} - 2\xi_{,t(j} \delta^i_{k)} + 3\eta^d_{,t} P^i_{,dkj} = 0 \quad (9)$$

$$L_\eta P^i_{jkd} - \xi_{,t} P^i_{jkd} + \xi_{,e} \delta^i_{(k} P^e_{,dj}) + \xi P^i_{,jkd,t} + 4\eta^e_{,t} P^i_{,jkde} - \xi_{,(j|k} \delta^i_{d)} = 0 \quad (10)$$

and the conditions due to the terms $\dot{x}^{j_1} \dots \dot{x}^{j_m}$ for $m > 4$ are given by the following general formula:

$$L_\eta P^i_{j_1 \dots j_m} + P^i_{j_1 \dots j_m, t} \xi + (2 - m) \xi_{,t} P^i_{j_1 \dots j_m} + \xi_{,r} (2 - (m - 1)) P^i_{j_1 \dots j_{m-1}} \delta^r_{j_m} + (m + 1) P^i_{j_1 \dots j_{m+1}} \eta^{j_{m+1}}_{,t} + \xi_{,j} P^j_{j_1 \dots j_{m-1}} \delta^i_{j_m} = 0. \quad (11)$$

We note the appearance of the term $L_\eta \Gamma^i_{jk}$ in these expressions.

Eqn (1) is obtained for $m = 0$, $P^i = F^i$ in which case the Lie symmetry conditions read:

$$L_\eta P^i + 2\xi_{,t} P^i + \xi P^i_{,t} + \eta^i_{,tt} = 0 \quad (12)$$

$$(\xi_{,k} \delta^i_j + 2\xi_{,j} \delta^i_k) P^k + 2\eta^i_{,t|j} - \xi_{,tt} \delta^i_k = 0 \quad (13)$$

$$L_\eta \Gamma^i_{jk} - 2\xi_{,t(j} \delta^i_{k)} = 0 \quad (14)$$

$$\xi_{,(j|k} \delta^i_{d)} = 0. \quad (15)$$

If $F^i = 0$ we obtain the Lie symmetry conditions for the geodesic equations (see [9]).

4 The autonomous dynamical system moving in a Riemannian space

We ‘solve’ the Lie symmetry conditions (12) - (15) for an autonomous dynamical system in the sense that we express them in terms of the collineations of the metric.

Equation (15) means that $\xi_{,j}$ is a gradient KV of g_{ij} . This implies that the metric g_{ij} is decomposable. Equation (14) means that η^i is a projective collineation of the metric with projective function $\xi_{,t}$. The remaining two equations are the constraint conditions, which relate the components ξ, n^i of the Lie symmetry vector with the vector $F^i(x^j)$. Equation (12) gives

$$(L_\eta g^{ij}) F_j + g^{ij} L_\eta F_j + 2\xi_{,t} g^{ij} F_j + \eta^i_{,tt} = 0. \quad (16)$$

This equation is an additional restriction for η^i because it relates it directly to the metric symmetries. Finally equation (13) gives

$$-\delta^i_j \xi_{,tt} + (\xi_{,j} \delta^i_k + 2\delta^i_j \xi_{,k}) F^k + 2\eta^i_{,tj} + 2\Gamma^i_{jk} \eta^k_{,t} = 0. \quad (17)$$

We conclude that the Lie symmetry equations are equations (16), (17) where $\xi(t, x)$ is a gradient KV of the metric g_{ij} and $\eta^i(t, x)$ is a special Projective collineation of the metric g_{ij} with projective function $\xi_{,t}$. We state the results in theorem 1 [9].

Theorem 1 *The Lie point symmetries of the system of equations of motion of an autonomous system under the action of the force $F^j(x^i)$ in a general Riemannian space with metric g_{ij} , namely*

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = F^i \quad (18)$$

are given in terms of the generators Y^i of the special projective algebra of the metric g_{ij} .

If the force F^i is derivable from a potential $V(x^i)$ so that the equations of motion follow from the standard Lagrangian

$$L(x^j, \dot{x}^j) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x^j) \quad (19)$$

with Hamiltonian

$$E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V(x^j) \quad (20)$$

the Noether conditions, are

$$V_{,k} \eta^k + V \xi_{,t} = -f_{,t} \quad (21)$$

$$\eta^i_{,t} g_{ij} - \xi_{,j} V = f_{,j} \quad (22)$$

$$L_{\eta} g_{ij} = 2 \left(\frac{1}{2} \xi_{,t} \right) g_{ij} \quad (23)$$

$$\xi_{,k} = 0. \quad (24)$$

Last equation implies $\xi = \xi(t)$ and reduces the system as follows

$$L_{\eta} g_{ij} = 2 \left(\frac{1}{2} \xi_{,t} \right) g_{ij} \quad (25)$$

$$V_{,k} \eta^k + V \xi_{,t} = -f_{,t} \quad (26)$$

$$\eta_{i,t} = f_{,i}. \quad (27)$$

Equation (25) implies that η^i is a conformal Killing vector of the metric provided $\xi_{,t} \neq 0$. Because g_{ij} is independent of t and $\xi = \xi(t)$ the η^i must be is a HV of the metric. This means that $\eta^i(t, x) = T(t) Y^i(x^j)$ where Y^i is a HV. If $\xi_{,t} = 0$ then η^i is a Killing vector of the metric. Equations (26), (27) are the constraint conditions, which the Noether symmetry and the potential must satisfy for the former to be admitted. These lead to the following theorem [9].

Theorem 2 *The Noether point symmetries of the Lagrangian (19) are generated from the homothetic algebra of the metric g_{ij} .*

More specifically, concerning the Noether symmetries, we have the following

All autonomous systems admit the Noether symmetry ∂_t whose Noether integral is the Hamiltonian E (20). For the rest of the Noether symmetries we consider the following cases

Case I Noether point symmetries generated by the homothetic algebra.

The Noether symmetry vector and the Noether function $G(t, x^k)$ are

$$\mathbf{X} = 2\psi_Y t \partial_t + Y^i \partial_i, \quad G(t, x^k) = pt \quad (28)$$

where, ψ_Y is the homothetic factor of Y^i ($\psi_Y = 0$ for a KV and 1 for the HV) and p is a constant, provided the potential satisfies the condition

$$\mathcal{L}_Y V + 2\psi_Y V + p = 0. \quad (29)$$

Case II Noether point symmetries generated by the gradient homothetic Lie algebra, i.e., both KVs and the HV are gradient.

In this case the Noether symmetry vector and the Noether function are

$$\mathbf{X} = 2\psi_Y \int T(t) dt \partial_t + T(t) H^i \partial_i, \quad G(t, x^k) = T_{,t} H(x^k) + p \int T dt \quad (30)$$

where, H^i is the gradient HV or a gradient KV, the function $T(t)$ is computed from the relation $T_{,tt} = mT$ where m is a constant and the potential satisfies the condition

$$\mathcal{L}_H V + 2\psi_Y V + mH + p = 0. \quad (31)$$

Concerning the Noether integrals we have the following result (not including the Hamiltonian)

Corollary 3 *The Noether integrals of Case I and Case II are respectively*

$$I_{C_I} = 2\psi_Y t E - g_{ij} Y^i \dot{x}^j + pt \quad (32)$$

$$I_{C_{II}} = 2\psi_Y \int T(t) dt E - g_{ij} H^i \dot{x}^j + T_{,t} H + p \int T dt \quad (33)$$

where, E is the Hamiltonian (20).

We remark that theorems 1 and 2 do not apply to generalized symmetries [15, 16].

In a number of recent papers [7, 17, 18, 19], the authors study the relation between the Noether symmetries of the geodesic Lagrangian

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \quad (34)$$

where $\dot{x}^a = \frac{dx^a}{ds}$ (s is an affine parameter along the geodesics) with the spacetime symmetries. They also make a conjecture concerning the relation between the Noether symmetries and the conformal algebra of spacetime and concentrate especially on conformally flat spacetimes. In [19] it is also claimed that the author has found new conserved quantities for spaces of different curvatures, which seem to be of nonnoetherian character. Obviously due to the above results (see also [8]) the conjecture/results in these papers should be revised and the word ‘conformal’ should be replaced with the word ‘homothetic’.

It would be of interest to examine if the above close relation of the Lie and the Noether symmetries of the second order ODEs of the form (6) with the collineations of the metric is possible to be carried over to some types of second order partial differential equations (PDEs). Although to this question it is not possible to give a global answer, due to the complexity of the study and the great variety of PDEs, it is still possible to give an answer of some generality which concerns many interesting and important cases. We will do this in the remaining sections.

5 The case of the second-order PDE's

In the attempt to establish a general relation between the Lie symmetries of a second order PDE and the collineations of a Riemannian space we derive the Lie symmetry conditions for a second order PDE of the form

$$A^{ij}u_{ij} - F(x^i, u, u_i) = 0 \quad (35)$$

and we consider the coefficients $A^{ij}(x, u)$ to be the components of a metric in a Riemannian space. According to the standard approach [1, 2, 3] the symmetry condition is

$$X^{[2]}(H) = \lambda H \quad (36)$$

where $\lambda(x^i, u, u_i)$ is a function to be determined. $X^{[2]}$ is the second prolongation of the Lie symmetry vector

$$X = \xi^i(x^i, u) \frac{\partial}{\partial x^i} + \eta(x^i, u) \frac{\partial}{\partial u} \quad (37)$$

given by the expression:

$$X^{[2]} = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \eta_i^{(1)} \frac{\partial}{\partial u_i} + \eta_{ij}^{(2)} \frac{\partial}{\partial u_{ij}} \quad (38)$$

where

$$\begin{aligned} \eta_i^{(1)} &= \frac{D\eta}{Dx^i} - u_j \frac{D\xi^j}{Dx^i} = \eta_{,i} + u_i \eta_u - \xi_{,i}^j u_j - u_i u_j \xi_{,u}^j \\ \eta_{ij}^{(2)} &= \frac{D\eta_i^{(1)}}{Dx^j} - u_{jk} \frac{D\xi^k}{Dx^j} = \eta_{ij} + (\eta_{ui} u_j + \eta_{uj} u_i) - \xi_{,ij}^k u_k + \eta_{uu} u_i u_j - (\xi_{,ui}^k u_j + \xi_{,uj}^k u_i) u_k \\ &\quad + \eta_u u_{ij} - (\xi_{,i}^k u_{jk} + \xi_{,j}^a u_{ik}) - (u_{ij} u_k + u_i u_{jk} + u_{ik} u_j) \xi_{,u}^k - u_i u_j u_k \xi_{uu}^k. \end{aligned}$$

The introduction of the function $\lambda(x^i, u, u_i)$ in (36) causes the variables x^i, u, u_i to be independent².

The symmetry condition $X^{[2]}(H) = \lambda H$ when applied to (35) gives:

$$A^{ij} \eta_{ij}^{(2)} + (X A^{ij}) u_{ij} - X^{[1]}(F) = \lambda (A^{ij} u_{ij} - F) \quad (39)$$

from which follows:

$$\begin{aligned} 0 &= A^{ij} \eta_{ij} - \eta_{,i} g^{ij} F_{,u_j} - X(F) + \lambda F \\ &\quad + 2A^{ij} \eta_{ui} u_j - A^{ij} \xi_{,ij}^a u_a - u_i \eta_u g^{ij} F_{,u_j} + \xi_{,i}^k u_k g^{ij} F_{,u_j} \\ &\quad + A^{ij} \eta_{uu} u_i u_j - 2A^{ij} \xi_{,uj}^k u_i u_k + u_i u_k \xi_{,u}^k g^{ij} F_{,u_j} \\ &\quad + A^{ij} \eta_u u_{ij} - 2A^{ij} \xi_{,i}^k u_{jk} + (\xi^k A_{,k}^{ij} + \eta A_{,u}^{ij}) u_{ij} - \lambda A^{ij} u_{ij} \\ &\quad - A^{ij} (u_{ij} u_a + u_i u_{ja} + u_{ia} u_j) \xi_{,u}^a - u_i u_j u_a A^{ij} \xi_{uu}^a. \end{aligned} \quad (40)$$

We note that we cannot deduce the symmetry conditions before we select a specific form for the function F . However we may determine the conditions which are due to the second derivative of u because in these terms no F terms are involved. This observation significantly reduces the complexity of the remaining symmetry condition. Following this observation we have the condition:

$$\begin{aligned} 0 &= A^{ij} \eta_u u_{ij} - A^{ij} (\xi_{,i}^k u_{ja} + \xi_{,j}^k u_{ik}) + (\xi^k A_{,k}^{ij} + \eta A_{,u}^{ij}) u_{ij} - \lambda A^{ij} u_{ij} \\ &\quad - A^{ij} (u_{ij} u_a + u_i u_{ja} + u_{ia} u_j) \xi_{,u}^a - u_i u_j u_a A^{ij} \xi_{uu}^a \end{aligned}$$

²See Ibragimov [11] p. 115

from which follow the equations:

$$\begin{aligned} A^{ij} (u_{ij}u_k + u_{jk}u_i + u_{ik}u_j) \xi_{.,u}^k &= 0 \\ A^{ij} \eta_u u_{ij} - A^{ij} (\xi_{.,i}^k u_{jk} + \xi_{.,j}^k u_{ik}) + (\xi^k A_{.,k}^{ij} + \eta A_{.,u}^{ij}) u_{ij} - \lambda A^{ij} u_{ij} &= 0 \\ A^{ij} \xi_{uu}^a &= 0. \end{aligned}$$

The first equation is written:

$$A^{ij} \xi_{.,u}^k + A^{kj} \xi_{.,u}^i + A^{ik} \xi_{.,u}^j = 0 \Leftrightarrow A^{(ij} \xi_{.,u}^{k)} = 0. \quad (41)$$

The second equation gives:

$$A^{ij} \eta_u + \eta A_{.,u}^{ij} + \xi^k A_{.,k}^{ij} - A^{kj} \xi_{.,k}^i - A^{ik} \xi_{.,k}^j - \lambda A^{ij} = 0. \quad (42)$$

and the last equation gives:

$$A^{ij} \xi_{uu}^k = 0. \quad (43)$$

It is straightforward to show³ that condition (41) implies

$$\xi_{.,u}^k = 0 \quad (44)$$

which is a well known result.

From the analysis so far we obtain the first result:

Proposition 4 *For all second-order PDEs of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$, for which at least one of the A^{ij} is $\neq 0$ the $\xi_{.,u}^i = 0$ or $\xi^i = \xi^i(x^j)$. Furthermore condition (43) is identically satisfied.*

There remains the third symmetry condition (42). We consider the following cases:

$i, j \neq 0$

We write(42) in an alternative form by considering A^{ij} to be a metric as follows:

$$L_{\xi^i \partial_i} A^{ij} = \lambda A^{ij} - (\eta A^{ij})_{.,u} \quad (45)$$

from which follows:

Proposition 5 *For all second-order PDEs of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$, for which $A^{ij}_{.,u} = 0$ i.e. $A^{ij} = A^{ij}(x^i)$, the vector $\xi^i \partial_i$ is a CKV of the metric A^{ij} with conformal factor $(\lambda - \eta_u)(x)$.*

Assuming⁴ $A^{tt} = A^{ti} = 0$ we have

- for $i = j = 0$ nothing
- for $i, j \neq 0$ gives (45) and
- for $i = 0, j \neq 0$ becomes:

$$\begin{aligned} A^{tj} \eta_u + \eta A_{.,u}^{tj} + \xi^k A_{.,k}^{tj} - A^{kj} \xi_{.,k}^t - A^{tk} \xi_{.,k}^j - \lambda A^{tj} &= 0 \Rightarrow \\ A^{kj} \xi_{.,k}^t &= 0 \end{aligned} \quad (46)$$

which leads to the following general result.

³We give a simple proof for $n = 2$ in Appendix A. A detailed and more general proof can be found in [20].

⁴The index t refers to the coordinate x^0 whenever it is involved.

Proposition 6 For all second-order PDEs of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$, for which A^{kj} is nondegenerate the $\xi_{.,k}^t = 0$, that is, $\xi^t = \xi^t(t)$.

Using that $\xi_{.,u}^i = 0$ when at least one of the $A_{ij} \neq 0$, the symmetry condition (40) is simplified as follows

$$\begin{aligned} 0 = & A^{ij}\eta_{ij} - \eta_{,i}A^{ij}F_{,u_j} - X(F) + \lambda F \\ & + 2A^{ij}\eta_{ui}u_j - A^{ij}\xi_{,ij}^a u_a - u_i\eta_u A^{ij}F_{,u_j} + \xi_{,i}^k u_k A^{ij}F_{,u_j} \\ & + A^{ij}\eta_{uu}u_iu_j + A^{ij}\eta_u u_{ij} - 2A^{ij}\xi_{.,i}^k u_{jk} \\ & + (\xi^k A_{,k}^{ij} + \eta A_{,u}^{ij})u_{ij} - \lambda A^{ij}u_{ij} \end{aligned} \quad (47)$$

which together with the condition (45) are the complete set of conditions for all second-order PDEs of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$, for which at least one of the $A_{ij} \neq 0$. This class of PDEs is quite general. This fact makes the above result very useful.

In order to continue we need to consider special forms for the function $F(x, u, u_i)$.

6 The Lie symmetry conditions for a linear function $F(x, u, u_i)$

We consider the function $F(x, u, u_i)$ to be linear in u_i , that is to be of the form

$$F(x, u, u_i) = B^k(x, u)u_k + f(x, u) \quad (48)$$

where $B^k(x, u)$, $f(x, u)$ are arbitrary functions of their arguments. In this case the PDE is of the form

$$A^{ij}u_{ij} - B^k(x, u)u_k - f(x, u) = 0. \quad (49)$$

The Lie symmetries of this type of PDEs have been studied previously by Ibragimov[11]. Assuming that at least one of the $A_{ij} \neq 0$ the Lie symmetry conditions are (47) and (45).

Replacing $F(x, u, u_1)$ in (47) we find⁵:

$$\begin{aligned} 0 = & A^{ij}\eta_{ij} - \eta_{,i}g^{ij}B_j - \xi^k f_{,k} - \eta f_{,u} + \lambda f \\ & + 2A^{ij}\eta_{ui}u_j - A^{ij}\xi_{,ij}^a u_a - u_i\eta_u g^{ij}B_j + \xi_{,i}^k u_k g^{ij}B_j + \lambda B^k u_k - \eta B_{,u}^k u_k - \xi^l B_{,l}^k u_k \\ & + A^{ik}\eta_{uu}u_iu_k \end{aligned} \quad (50)$$

$$+ A^{ij}\eta_u u_{ij} - 2A^{kj}\xi_{.,k}^i u_{ji} + (\xi^k A_{,k}^{ij} + \eta A_{,u}^{ij})u_{ij} - \lambda A^{ij}u_{ij} \quad (51)$$

from which follow the equations:

$$A^{ij}\eta_{ij} - \eta_{,i}B^i - \xi^k f_{,k} - \eta f_{,u} + \lambda f = 0 \quad (52)$$

$$-2A^{ik}\eta_{ui} + A^{ij}\xi_{,ij}^k + \eta_u B^k - \xi_{,i}^k B^i + \xi^i B_{,i}^k - \lambda B^k + \eta B_{,u}^k = 0 \quad (53)$$

$$A^{ik}\eta_{uu} = 0. \quad (54)$$

⁵We ignore the terms with u_{ij} because we have already use them to obtain condition (45). Indeed it is easy to see that these terms give $A^{ij}\eta_u - 2A^{ij}\xi_{.,i}^k + \xi^k A_{,k}^{ij} + \eta A_{,u}^{ij} - \lambda A^{ij} = 0$ which is precisely condition (45).

Equation (54) gives (because at least one $A^{ik} \neq 0$) :

$$\eta = a(x^i)u + b(x^i). \quad (55)$$

Equation (53) gives

$$-2A^{ik}a_{,i} + aB^k + auB_{,u}^k + A^{ij}\xi_{,ij}^k - \xi_{,i}^k B^i + \xi^i B_{,i}^k - \lambda B^k + bB_{,u}^k = 0.$$

We summarize the above results as follows.

The Lie symmetry conditions for the second order PDEs of the form

$$A^{ij}u_{ij} - B^k(x, u)u_k - f(x, u) = 0 \quad (56)$$

where at least one of the $A_{ij} \neq 0$ are:

$$A^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})B^i - \xi^k f_{,k} - au f_{,u} - bf_{,u} + \lambda f = 0 \quad (57)$$

$$A^{ij}\xi_{,ij}^k - 2A^{ik}a_{,i} + aB^k + auB_{,u}^k - \xi_{,i}^k B^i + \xi^i B_{,i}^k - \lambda B^k + bB_{,u}^k = 0 \quad (58)$$

$$L_{\xi^i \partial_i} A^{ij} = (\lambda - a)A^{ij} - \eta A^{ij}_{,u} \quad (59)$$

$$\eta = a(x^i)u + b(x^i) \quad (60)$$

$$\xi_{,u}^k = 0 \Leftrightarrow \xi^k(x^i). \quad (61)$$

We note that for all second order PDEs of the form $A^{ij}u_{ij} - B^k(x, u)u_k - f(x, u) = 0$ for which $A^{ij}_{,u} = 0$ i.e. $A^{ij}(x^i)$, the $\xi^i(x^j)$ is a CKV of the metric A^{ij} . Also in this case $\lambda(x^i)$. This result establishes the relation between the Lie symmetries of this type of PDEs with the collineations of the metric defined by the coefficients A_{ij} .

Furthermore in case the coordinates are t, x^i (where $i = 1, \dots, n$) $A^{tt} = A^{tx^i} = 0$ and A^{ij} is a nondegenerate metric we have that

$$\xi_{,i}^t = 0 \Leftrightarrow \xi^t(t). \quad (62)$$

These symmetry relations coincide with those given in [11].

Finally note that equation (58) can be written:

$$A^{ij}\xi_{,ij}^k - 2A^{ik}a_{,i} + [\xi, B]^k + (a - \lambda)B^k + (au + b)B_{,u}^k = 0. \quad (63)$$

Having derived the Lie symmetry conditions for the type of PDEs of the form $A^{ij}u_{ij} - B^k(x, u)u_k - f(x, u) = 0$ we continue with the computation of the Lie symmetries of some important PDEs of this form.

Before we proceed, we state two Lemmas which will be used in the discussion of the examples.

Lemma 7 *a. In flat space (in which $\Gamma_{jk}^i = 0$) the following identity holds:*

$$L_{\xi} \Gamma_{ij}^k = \xi_{,ij}^k. \quad (64)$$

b. For a general metric g_{ij} satisfying the condition $L_{\xi^i \partial_i} g_{ij} = -(\lambda - a)g_{ij}$ the following relation holds:

$$g^{jk} L_{\xi} \Gamma_{jk}^i = g^{jk} \xi_{,jk}^i + \Gamma_{,l}^i \xi^l - \xi_{,l}^i \Gamma^l + (a - \lambda) \Gamma^i. \quad (65)$$

Proof

Using the formula

$$L_\xi \Gamma_{.jk}^i = \Gamma_{.jk,l}^i \xi^l + \xi_{,jk}^i - \xi_{,l}^i \Gamma_{.jk}^l + \xi_{,j}^s \Gamma_{.sk}^i + \xi_{,k}^s \Gamma_{.sj}^i$$

we have:

$$\begin{aligned} g^{jk} L_\xi \Gamma_{.jk}^i &= \Gamma_{,l}^i \xi^l + g^{jk} \xi_{,jk}^i - g^{jk} \xi_{,l}^i \Gamma_{.jk}^l - \xi_{,l}^i \Gamma^l + 2g^{jk} \xi_{,j}^s \Gamma_{.sk}^i \\ &= \Gamma_{,l}^i \xi^l + g^{jk} \xi_{,jk}^i - \xi_{,l}^i \Gamma^l \\ &\quad - [g^{jl} \xi_{,l}^k + g^{kl} \xi_{,l}^j - (\lambda - a) g^{jk}] \Gamma_{.jk}^i + 2g^{jk} \xi_{,j}^s \Gamma_{.sk}^i \\ &= \Gamma_{,l}^i \xi^l + g^{jk} \xi_{,jk}^i - \xi_{,l}^i \Gamma^l + 2(g^{jl} \xi_{,j}^k \Gamma_{.kl}^i - g^{lj} \xi_{,j}^k \Gamma_{.kl}^i) - (\lambda - a) \Gamma^i \\ &= \Gamma_{,l}^i \xi^l - \xi_{,l}^i \Gamma^l + g^{jk} \xi_{,jk}^i + (a - \lambda) \Gamma^i \end{aligned}$$

Lemma 8 Assume that the vector ξ^i is a CKV of the metric g_{ij} with conformal factor $-(\lambda - a)$ i.e. $L_{\xi^i \partial_i} g_{ij} = -(\lambda - a) g_{ij}$. Then the following statement is true:

$$g^{jk} L_\xi \Gamma_{.jk}^i = \frac{2-n}{2} (a - \lambda)^{,i} \quad (66)$$

where $n = g^{jk} g_{kj}$ the dimension of the space.

Proof

Using the identity:

$$L_\xi \Gamma_{.jk}^i = \frac{1}{2} g^{ir} [\nabla_k L_\xi g_{jr} + \nabla_j L_\xi g_{kr} - \nabla_r L_\xi g_{kj}] \quad (67)$$

and replacing $L_\xi g_{ij} = (a - \lambda) g_{ij}$ we find:

$$\begin{aligned} L_\xi \Gamma_{.jk}^i &= \frac{1}{2} g^{ir} [(a - \lambda)_{,k} g_{jr} + (a - \lambda)_{,j} g_{kr} - (a - \lambda)_{,r} g_{kj}] \\ &= \frac{1}{2} [(a - \lambda)_{,k} \delta_j^i + (a - \lambda)_{,j} \delta_k^i - g^{ir} (a - \lambda)_{,r} g_{kj}]. \end{aligned}$$

Contracting with g^{jk} we obtain the required result.

7 Applications

7.1 The wave equation for an inhomogeneous medium

In order to show how the above considerations are applied in practice we consider the wave equation for an inhomogeneous medium in flat 2d Newtonian space:

$$c^2(x^1) u_{11} - u_{22} = 0. \quad (68)$$

In this case we have:

$$A_{11} = c^{-2}(x^1), \quad A_{22} = -1, \quad A_{12} = 0; \quad B^i = 0; \quad f = 0.$$

The symmetry conditions (57) - (61) become:

$$A^{ij} (a_{ij} u + b_{ij}) = 0 \quad (69)$$

$$A^{ij}\xi^k_{,ij} - 2A^{ik}a_i = 0 \quad (70)$$

$$L_{\xi^i \partial_i} A^{ij} = (\lambda - a)A^{ij} \quad (71)$$

$$\eta = a(x^i)u + b(x^i) \quad (72)$$

$$\xi^k(x). \quad (73)$$

The vector ξ^i is a CKV of the metric $A_{ij} = \text{diag}(c^{-2}(x^1), -1)$ with conformal factor $-(\lambda - a)$. This is a⁶ nondegenerate 2-d metric which is conformally flat, therefore if we find the conformal factor we will have the solution ξ^i and the function $a(x^i)$.

We take now the metric to be the A^{ij} . Then according to Lemma 7 we have (where Γ^i_{jk} are the Γ' s of the metric A_{ij}) :

$$\begin{aligned} A^{jk}L_{\xi}\Gamma^i_{jk} &= \Gamma^i_{,l}\xi^l - \xi^i_{,l}\Gamma^l + A^{jk}\xi^i_{,jk} - (\lambda - a)\Gamma^i_{,l} \Rightarrow \\ A^{jk}\xi^i_{,jk} &= A^{jk}L_{\xi}\Gamma^i_{jk} + \xi^i_{,l}\Gamma^l - \Gamma^i_{,l}\xi^l + (\lambda - a)\Gamma^i_{,l} \end{aligned} \quad (74)$$

Then the Lie symmetry condition (70) $A^{ij}\xi^k_{,ij} = 2A^{ik}a_i$ is written:

$$A^{ij}(L_{\xi}\Gamma^k_{ij} - 2\delta^k_j a_{,i}) = \Gamma^i_{,l}\xi^l - \xi^i_{,l}\Gamma^l - (\lambda - a)\Gamma^i_{,l}. \quad (75)$$

For the metric $A_{ij} = \text{diag}(-c^2(x^1), -1)$ we compute $\Gamma^i = \Gamma^i_{,l} = 0$ hence the last equation becomes:

$$A^{ij}(L_{\xi}\Gamma^k_{ij} - 2\delta^k_j a_{,i}) = 0. \quad (76)$$

Because the metric A_{ij} is nondegenerate this implies

$$L_{\xi}\Gamma^k_{ij} = 2\delta^k_{(j}a_{,i)} \quad (77)$$

which means that ξ^i is a projective vector of the metric A_{ij} with projection function a . But ξ^i is also a CKV of the same metric with conformal factor $a - \lambda$ therefore ξ^i must be a HV of the metric A_{ij} . This implies

$$a = c_1 \text{ a constant}$$

and

$$-\lambda + c_1 = c_2 \Rightarrow \lambda = c_1 - c_2$$

where c_2 is the homothetic factor. From the remaining condition (69) we have $A^{ij}b_{,ij} = 0$ that is, the function b is a solution of the original wave equation (68). We conclude that the Lie symmetry vector is (see also [1] p. 182):

$$X = \xi^i \partial_i + (c_1 u + b) \partial_u \quad (78)$$

where ξ^i is a HV (not necessarily proper) of the metric A_{ij} and $b(x^i)$ is a solution of the wave equation. The vector $\xi^i \partial_i$ is the sum $a_i KV^i + a_{HV} HV$ where a_i, a_{HV} are constants and KV^i, HV are any of the KVs and the HV (if it exists) of the metric A_{ij} .

In the following section we consider a new example which is the heat conduction equation with a flux in an n -dimensional Riemannian space.

⁶Because $\det A_{ij} = -c^2(x^1) \neq 0$ the inverse of A^{ij} exists.

7.2 The heat conduction equation with a flux in a Riemannian space

The heat equation with a flux in an n -dimensional Riemannian space with metric g_{ij} is:

$$H(u) = q(t, x^i, u) \quad (79)$$

where

$$H(u) := g^{ij}u_{ij} - \Gamma^i u_i - u_t.$$

The term q indicates that the system exchanges energy with the environment. In this case the Lie point symmetry vector is:

$$\mathbf{X} = \xi^i(x^j, u) \partial_i + \eta(x^j, u) \partial_u$$

where $a = t, i$. For this equation we have:

$$A^{tt} = 0, A^{ti} = 0, A^{ij} = g^{ij}, B^i = \Gamma^i(t, x^i), B^t = 1, f(x, u) = q(t, x^k, u).$$

For this PDE the symmetry conditions (57) - (62) become :

$$\eta = a(t, x^i)u + b(t, x^i) \quad (80)$$

$$\xi^t = \xi^t(t) \quad (81)$$

$$g^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})\Gamma^i - (a_{,t}u + b_{,t}) + \lambda q = \xi^t q_{,t} + \xi^k q_{,k} + \eta q_{,u} \quad (82)$$

$$g^{ij}\xi_{,ij}^k - 2g^{ik}a_{,i} + a\Gamma^k - \xi_{,i}^k\Gamma^i + \xi^i\Gamma_{,i}^k - \lambda\Gamma^k = 0 \quad (83)$$

$$L_{\xi^i \partial_i} g_{ij} = (a - \lambda)g_{ij}. \quad (84)$$

The solution of the symmetry conditions is summarized in Theorem 9. The proof of the theorem is given in Appendix B.

Theorem 9 *The Lie point symmetries of the heat equation with flux i.e.*

$$g^{ij}u_{ij} - \Gamma^i u_i - u_t = q(t, x, u) \quad (85)$$

in a n -dimensional Riemannian space with metric g_{ij} are constructed from the homothetic algebra of the metric as follows:

a. Y^i is a nongradient HV/KV.

The Lie point symmetry is

$$X = (2c_2\psi t + c_1) \partial_t + c_2 Y^i \partial_i + (a(t)u + b(t, x)) \partial_u \quad (86)$$

where $a(t), b(t, x^k), q(t, x^k, u)$ must satisfy the constraint equation

$$-a_t u + H(b) - (au + b)q_{,u} + aq - (2\psi c_2 q t + c_1 q)_t - c_2 q_{,i} Y^i = 0. \quad (87)$$

b. $Y^i = S^i$ is a gradient HV/KV.

The Lie point symmetry is

$$X = \left(2\psi \int T dt + c_1 \right) \partial_t + T S^i \partial_i + \left(\left(-\frac{1}{2} T_{,t} S + F(t) \right) u + b(t, x) \right) \partial_u \quad (88)$$

where $F(t), T(t), b(t, x^k), q(t, x^k, u)$ must satisfy the constraint equation

$$0 = \left(-\frac{1}{2}T_{,t}\psi + \frac{1}{2}T_{,tt}S - F_{,t} \right) u + H(b) + \left(\left(-\frac{1}{2}T_{,t}S + F \right) u + b \right) q_{,u} + \left(-\frac{1}{2}T_{,t}S + F \right) q - \left(2\psi q \int T dt + c_1 q \right)_t - T q_{,i} S^{,i}. \quad (89)$$

We apply theorem 9 for special forms of the function $q(t, x, u)$.

7.2.1 The homogeneous heat equation i.e. $q(t, x, u) = 0$

In this case we have the following results

Theorem 10 *The Lie point symmetries of the homogeneous heat conduction equation in an n -dimensional Riemannian space*

$$g^{ij}u_{ij} - \Gamma^i u_i - u_t = 0 \quad (90)$$

are constructed from the homothetic algebra of the metric g_{ij} as follows:

(a) If Y^i is a nongradient HV/KV of the metric g_{ij} , the Lie point symmetry is

$$X = (2\psi c_1 t + c_2) \partial_t + c_1 Y^i \partial_i + (a_0 u + b(t, x^i)) \partial_u \quad (91)$$

where c_1, c_2, a_0 are constants and $b(t, x^i)$ is a solution of the homogeneous heat equation.

(c) If $Y^i = S^{,i}$ is a gradient HV/KV of the metric g_{ij} the Lie point symmetry is

$$X = (c_3 \psi t^2 + c_4 t + c_5) \partial_t + (c_3 t + c_4) S^i \partial_i + \left(-\frac{c_3}{2} S - \frac{c_3}{2} n \psi t + c_5 \right) u \partial_u + b(t, x^i) \partial_u \quad (92)$$

where c_3, c_4, c_5 are constants and $b(t, x^i)$ is a solution of the homogeneous heat equation.

In order to compare the above result with the existing results in the literature we consider the heat equation in a Euclidian space of dimension n . Then in Cartesian coordinates $g_{ij} = \delta_{ij}$, $\Gamma^i = 0$ and the heat equation is:

$$\delta^{ij}u_{ij} - u_t = 0. \quad (93)$$

The homothetic algebra of the space consists of the n gradient KVs ∂_i with generating functions x^i , the $\frac{n(n-1)}{2}$ nongradient KVs X_{IJ} which are the rotations and a gradient HV H^i with gradient function $H = R\partial_R$. According to theorem 10 the Lie symmetries of the heat equation in the Euclidian n dimensional space are (we may take $\psi = 1$)

$$X = [c_3 \psi t^2 + (c_4 + 2\psi c_1)t + c_5 + c_2] \partial_t + [c_1 Y^i + (c_3 t + c_4) S^i] \partial_i + \left[\left(a_0 + \frac{c_3}{2} S + \frac{c_3}{2} n \psi t - c_5 \right) u + b(t, x^i) \right] \partial_u. \quad (94)$$

This result agrees with the results of [3] p. 158.

Next we consider the de Sitter spacetime (a four dimensional space of constant curvature and Lorentzian character) whose metric is:

$$ds^2 = \frac{(-d\tau^2 + dx^2 + dy^2 + dz^2)}{\left(1 + \frac{K}{4}(-\tau^2 + x^2 + y^2 + z^2)\right)^2}$$

It is known that the homothetic algebra of this space consists of the ten KVs

$$\begin{aligned}
X_1 &= (-x\tau) \partial_\tau + \left(\frac{(-\tau^2 - x^2 + y^2 + z^2)}{2} - \frac{2}{K} \right) \partial_x + (-yx) \partial_y + (-zx) \partial_z \\
X_2 &= (y\tau) \partial_\tau + (yx) \partial_x + \left(\frac{(-x^2 - z^2 + y^2 + \tau^2)}{2} + \frac{2}{K} \right) \partial_y + (yz) \partial_z \\
X_3 &= (z\tau) \partial_\tau + (zx) \partial_x + (zy) \partial_y + \left(\frac{(-x^2 - y^2 + z^2 + \tau^2)}{2} + \frac{2}{K} \right) \partial_z \\
X_4 &= \left(\frac{(x^2 + y^2 + z^2 + \tau^2)}{2} - \frac{2}{K} \right) \partial_\tau + (\tau x) \partial_x + (\tau y) \partial_y + (\tau z) \partial_z \\
X_5 &= x\partial_\tau + \tau\partial_x, \quad X_6 = y\partial_\tau + \tau\partial_y, \quad X_7 = z\partial_\tau + \tau\partial_z, \quad X_8 = y\partial_x - x\partial_y \\
X_9 &= z\partial_x - x\partial_z, \quad X_{10} = z\partial_y - y\partial_z
\end{aligned}$$

all of which are nongradient. According to Theorem 10 the Lie symmetries of the heat equation in de Sitter space are

$$\partial_t + \sum_{A=1}^{10} c_A X_A + (a_0 u + b(x, u)) \partial_u.$$

From Theorem 10 we have the following additional results.

Corollary 11 *The one dimensional homogenous heat equation admits a maximum number of seven Lie symmetries (modulo a solution of the heat equation).*

Proof. The homothetic group of a 1-dimensional metric $ds^2 = g^2(x) dx^2$ consists of one gradient KV (the $\frac{1}{g(x)} \partial_x$) and one gradient HV ($\frac{1}{g(x)} \int g(x) dx \partial_x$). According to theorem 10 from the KV we have two Lie symmetries and from the gradient HV another two Lie symmetries. To these we have to add the two Lie symmetries $X = a_0 u \partial_u + b(t, x^i) \partial_u$ and the trivial Lie symmetry ∂_t where $b(t, x^i)$ is a solution of the heat equation. ■

Corollary 12 *The homogeneous heat equation in a space of constant curvature of dimension n has at most $(n+3) + \frac{1}{2}n(n-1)$ (modulo a solution of the heat equation).*

Proof. A space of constant curvature of dimension n admits $n + \frac{1}{2}n(n-1)$ nongradient KVs To these we have to add the Lie symmetries $X = c\partial_t + a_0 u \partial_u + b(t, x^i) \partial_u$. ■

Corollary 13 *The heat conduction equation in a space of dimension n admits at most $\frac{1}{2}n(n+3) + 5$ Lie symmetries (modulo a solution of the heat equation) and if this is the case the space is flat.*

Proof. The space with the maximum homothetic algebra is the flat space which admits n gradient KVs, $\frac{1}{2}n(n-1)$ nongradient KVs and one gradient HV. Therefore from Case 1, of the theorem we have $(n+1) + \frac{1}{2}n(n-1)$ Lie symmetries. From Case 2. we have another $(n+1)$ Lie symmetries and to these we have to add the Lie symmetries $X = c_1 \partial_t + a_0 u \partial_u + b(t, x^i) \partial_u$ where $b(t, x^i)$ is a solution of the heat equation. The set of all these symmetries is $1 + 2n + \frac{1}{2}n(n-1) + 2 + 1 + 1 = \frac{1}{2}n(n+3) + 5$ [11]. ■

7.2.2 Case $q(t, x, u) = q(u)$

In this case we have the following result:

Theorem 14 *The Lie symmetries of the heat equation with conduction $q(u)$ in a n dimensional Riemannian space*

$$g^{ij}u_{ij} - \Gamma^i u_i - u_t = q(u) \quad (95)$$

are constructed from the homothetic algebra of the metric as follows.

a. Y^i is a HV/KV

The Lie point symmetry is

$$X = (2c\psi t + c_1) \partial_t + cY^i \partial_i + (a(t)u + b(t, x)) \partial_u \quad (96)$$

where the functions $a(t)$, $b(t, x)$ and $q(u)$ satisfy the condition

$$-a_t u + H(b) - (au + b)q_{,u} + (a - 2\psi c)q = 0. \quad (97)$$

b. $Y^i = S^{,i}$ is a gradient HV/KV

The Lie point symmetry is

$$X = \left(2\psi \int T dt + c_1\right) \partial_t + TS^{,i} \partial_i + \left(\left(-\frac{1}{2}T_{,t}S + F(t)\right)u + b(t, x)\right) \partial_u \quad (98)$$

where $b(t, x)$ is a solution of the homogeneous heat equation, the functions $T(t)$, $F(t)$ and the flux $q(u)$ satisfy the equation:

$$\left(-\frac{1}{2}T_{,t}\psi + \frac{1}{2}T_{,tt}S - F_{,t}\right)u + H(b) - \left(\left(-\frac{1}{2}T_{,t}S + F\right)u + b\right)q_{,u} + \left(-\frac{1}{2}T_{,t}S + F\right)q - 2\psi qT = 0 \quad (99)$$

For various cases of $q(u)$ we obtain the results of the following table.

$q(u)$	Lie Symmetry vector
$q_0 u$	$(\psi T_0 t^2 + 2c\psi t + c_1) \partial_t + (cY^i + T_0 t S^{,i}) \partial_i +$ $+ \left([-2\psi c q_0 t + a_0 + T_0 \left(-\frac{1}{2}S - \psi q_0 t^2 - \frac{1}{2}t\right)] u + b(t, x)\right) \partial_u$ where $H(b) - b q_0 = 0$
$q_0 u^n$	$(2c\psi t + c_1) \partial_t + cY^i \partial_i + \left(\frac{2\psi c}{1-n}u\right) \partial_u$
$u \ln u$	$c_1 \partial_t + (Y^i + T_0 e^{-t} K^{,i}) \partial_i + (a_0 e^{-t} u) \partial_u, K^{,i}$
e^u	$(2c\psi t + c_1) \partial_t + cY^i \partial_i + (-2\psi c) \partial_u$

where Y^i is a HV/KV, $S^{,i}$ is a gradient HV/KV and $K^{,i}$ is a gradient KV.

8 Conclusion

The main result of this work is Proposition 5 which states that the Lie symmetries of the PDEs of the form (36) are obtained from the conformal Killing vectors of the metric defined by the coefficients A_{ij} , provided $A_{ij,u} = 0$.

This result is quite general and covers many well known and important PDEs of Physics. The geometrization of the Lie symmetries and their association with the collineations of the metric, dissociates their determination from the dimension of the space, because the collineations of the metric depend (in general) on the type of the metric and not on the dimensions of the space where the metric resides. Furthermore, this association provides a wealth of results of Differential Geometry on collineations which is possible to be used in the determination of the Lie symmetries.

We have applied the above theoretical results to two cases. The first case concerns the two dimensional wave equation in an inhomogeneous medium and shows the application of the general results in practice. The second example concerns the determination of the Lie symmetry vectors of the heat conduction equation with a flux in a Riemannian space, a problem which has not been considered before in the literature. We proved that the Lie symmetry algebra of this PDE is generated from the homothetic algebra of the metric. We specialized the equation to the homogeneous heat conduction equation and regained the existing results for the Newtonian case. Finally it can be shown that the Lie symmetries of the Poisson equation in a Riemannian space which have been computed in [12] are obtained from the present formalism in a straightforward manner.

Appendix A

We prove the statement for $n = 2$. The generalization to any n is straightforward. For a general proof see [20]. We consider A^{ij} as a matrix and assume that the inverse of this matrix exists. We denote the inverse matrix with B_{ij} and we get from (41):

$$\begin{aligned} B_{ij}A^{ij}\xi_{.,u}^k + B_{ij}A^{kj}\xi_{.,u}^i + B_{ij}A^{ik}\xi_{.,u}^j &= 0 \\ 2\xi_{.,u}^k + \delta_i^k\xi_{.,u}^i + \delta_j^k\xi_{.,u}^j &= 0 \Rightarrow \\ \xi_{.,u}^k &= 0. \end{aligned} \tag{100}$$

Now assume that the matrix A^{ij} does not have an inverse. Then we consider $n = 2$ and write:

$$[A^{ij}] = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \Rightarrow \det A^{ij} = A^{11}A^{22} - (A^{12})^2 = 0$$

where at least one of the $A^{ij} \neq 0$. Assume $A^{11} \neq 0$. Then equation (41) for $i = j = k = 1$ gives

$$3A^{11}\xi_{.,u}^1 = 0 \Rightarrow \xi_{.,u}^1 = 0.$$

The same equation for $i = j = k = 2$ gives

$$3A^{22}\xi_{.,u}^2 = 0$$

therefore either $\xi_{.,u}^2 = 0$ or $A^{22} = 0$. If $A^{22} = 0$ then from the condition $\det A^{ij} = 0$ we have $A^{12} = 0$ hence $A_{ij} = 0$ which we do not assume. Therefore $\xi_{.,u}^2 = 0$.

We consider now equations $i = j \neq k$ and find:

$$A^{ii}\xi_{.,u}^k + A^{ki}\xi_{.,u}^i + A^{ik}\xi_{.,u}^i = 0.$$

Because $i \neq k$ this gives $A^{ii}\xi_{.,u}^k = 0$ and because we have assumed $A^{11} \neq 0$ it follows $\xi_{.,u}^2 = 0$. Therefore again we find $\xi_{.,u}^k = 0$.

9 Appendix B

Condition (84) means that ξ^i is a CKV of the metric g_{ij} with conformal factor $a(t, x^k) - \lambda(t, x^k)$.

Condition (83) implies $\xi^k = T(t) Y^k(x^j)$ where Y^i is a HV with conformal factor ψ , that is, we have:

$$L_{Y^i} g_{ij} = 2\psi g_{ij} \quad , \quad \psi = \text{constant.}$$

and

$$\xi_{,t}^t = a - \lambda$$

from which follow

$$\xi^t(t) = 2\psi \int T dt. \quad (101)$$

$$-2g^{ik} a_{,i} + T_{,t} Y^k = 0. \quad (102)$$

Condition (82) becomes:

$$\begin{aligned} H(a)u + H(b) + (a - \xi_{,t}^t)q &= \xi^t q_{,t} + T(t) Y^k q_{,k} + \eta q_{,u} \Rightarrow \\ H(a)u + H(b) - (au + b)q_{,u} + aq - (\xi^t q)_{,t} - T(t) Y^k q_{,k} &= 0 \\ H(a)u + H(b) - (au + b)q_{,u} + aq - \left(2\psi q \int T dt\right)_t - T q_{,i} Y^i &= 0. \end{aligned} \quad (103)$$

We consider the following cases:

Case 1 Y^k is a HV/KV

From (102) we have that $T_{,t} = 0 \rightarrow T(t) = c_2$ and $a_{,i} = 0 \rightarrow a(t, x^k) = a(t)$. Then (103) becomes

$$-a_t u + H(b) - (au + b)q_{,u} + aq - (2\psi c_2 q t + c_1 q)_t - c_2 q_{,i} Y^i = 0 \quad (104)$$

Case 2 Y^k is a gradient HV/KV, that is $Y^k = S^{,k}$

From (102) we have

$$a(t, x^k) = -\frac{1}{2} T_{,t} S + F(t). \quad (105)$$

Replacing in (103) we find the constraint equation:

$$\begin{aligned} 0 &= \left(-\frac{1}{2} T_{,t} \psi + \frac{1}{2} T_{,tt} S - F_{,t}\right) u + H(b) + \\ &\quad - \left(\left(-\frac{1}{2} T_{,t} S + F\right) u + b\right) q_{,u} + \left(-\frac{1}{2} T_{,t} S + F\right) q - \left(2\psi q \int T dt + c_1 q\right)_t - T q_{,i} S^{,i}. \end{aligned} \quad (106)$$

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